ON THE HERZOG-SCHÖNHEIM CONJECTURE
FOR UNIFORM COVERS OF GROUPS

ZHI-WEI SUN

Department of Mathematics, Nanjing University
Nanjing 210093, The People's Republic of China
E-mail: `zwsun@nju.edu.cn`
Homepage: `http://pweb.nju.edu.cn/zwsun`

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ABSTRACT. Let G be any group and $a_1 G_1, \ldots, a_k G_k$ $(k > 1)$ be left cosets in G. In 1974 Herzog and Schönheim conjectured that if $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ is a partition of G then the (finite) indices $n_1 = [G : G_1], \ldots, n_k = [G : G_k]$ cannot be pairwise distinct. In this paper we show that if $\mathcal{A}$ covers all the elements of G the same number of times and $G_1, \ldots, G_k$ are subnormal subgroups of G not all equal to G, then $M = \max_{1 \leqslant j \leqslant k} |\{1 \leqslant i \leqslant k : n_i = n_j\}|$ is not less than the smallest prime divisor of $n_1 \cdots n_k$, moreover $\min_{1 \leqslant i \leqslant k} \log n_i = O(M \log^2 M)$ where the O-constant is absolute.

1. INTRODUCTION

Let G be a (multiplicative) group. As usual we use e to denote the identity element of G. A left coset of a subgroup H in G is in the form $aH = \{ah : h \in H\}$ where $a \in G$. For a finite system

$$\mathcal{A} = \{a_i G_i\}_{i=1}^k \tag{1.1}$$

of left cosets in G, if

$$w_{\mathcal{A}}(x) = |\{1 \leqslant i \leqslant k : x \in a_i G_i\}| \tag{1.2}$$

does not depend on $x \in G$ then we call (1.1) a *uniform cover* of G. Only in the case $G_1 = \cdots = G_k = G$, (1.1) is regarded as a *trivial* uniform cover of G. If $w_{\mathcal{A}}(x) = 1$

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for all $x \in G$ then we call (1.1) a *disjoint cover* (or *partition*) of G. A uniform cover may have no disjoint subcover (cf. [Gu]).

Any infinite cyclic group is isomorphic to the additive group $\mathbb{Z}$ of the integers. The subgroups of $\mathbb{Z}$ different from $\{0\}$ are in the form $n\mathbb{Z} = \{nx : x \in \mathbb{Z}\}$ where $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. For any positive integer n, the index of $n\mathbb{Z}$ in $\mathbb{Z}$ is n and a coset of $n\mathbb{Z}$ in $\mathbb{Z}$ is just a residue class

$$a + n\mathbb{Z} = \{x \in \mathbb{Z} : x \equiv a \pmod{n}\} \qquad \text{where } a \in \mathbb{Z}.$$

A finite system

$$A = \{a_i + n_i\mathbb{Z}\}_{i=1}^k \qquad (n_1 \leqslant \cdots \leqslant n_k) \tag{1.3}$$

of residue classes is called a cover of $\mathbb{Z}$ if $\bigcup_{i=1}^k a_i + n_i\mathbb{Z} = \mathbb{Z}$. Such covers were introduced by P. Erdős ([E1]) in the early 1930's, they have many surprising applications (see, e.g. [Cr], [Gr], [Sc], [Su7], [Su9] and [Su10]). Soon after his invention of the concept of cover of $\mathbb{Z}$, Erdős made a conjecture that (1.3) cannot be a partition of $\mathbb{Z}$ if $1 < n_1 < \cdots < n_k$. This was confirmed by H. Davenport, L. Mirsky, D. Newman and R. Rado (see [E2] and [NZ]) who used analysis to show that if (1.3) forms a partition of $\mathbb{Z}$ with $k > 1$ then $n_{k-1} = n_k$. The reader may consult [Su4], [Su5] and [Su6] for progress on uniform covers of $\mathbb{Z}$.

In the 1950's B. H. Neumann ([N1], [N2]) studied groups as unions of cosets of subgroups while he didn't know number-theoretic research on covers of $\mathbb{Z}$. A basic result of Neumann [N1] is as follows: If (1.1) forms a cover of a group G by left cosets but none of its proper subsystems does, then $[G : \bigcap_{i=1}^k G_i] \leqslant c_k$ where c_k is a constant depending on k. In 1987 M. J. Tomkinson [To] strengthened the Neumann result by showing that we can take $c_k = k!$. By Corollary 1 of the author [Su1], for any uniform cover (1.1) of a group G we also have $[G : \bigcap_{i=1}^k G_i] \leqslant k!$.

In 1958 S. K. Stein [St] suggested that investigations on covers of $\mathbb{Z}$ should be carried out on covers of abstract groups. In 1974 M. Herzog and J. Schönheim [HS] proposed the following generalization of Erdős' conjecture.

Herzog–Schönheim Conjecture. *Let* (1.1) *be a partition of a group G into $k > 1$ left cosets. Then at least two of the finite indices $[G : G_1], \ldots, [G : G_k]$ are equal.*

M. M. Parmenter [Pa] and R. Brandl [Br] partially told us when all the subgroups G_i in a partition (1.1) of group G are equal or conjugate in G. The Herzog-Schönheim conjecture can be extended to uniform covers of groups.

A finite group G is said to be *pyramidal* if it contains a chain $\{e\} = H_0 \subset H_1 \subset \cdots \subset H_n = G$ of subgroups such that $[H_1 : H_0] \geqslant \cdots \geqslant [H_n : H_{n-1}]$ are primes in non-ascending order. In such a chain H_{i-1} is normal in H_i since $[H_i : H_{i-1}]$ is the smallest prime dividing $|H_i|$ (see [Ro, 4.18]), therefore the chain of H's forms a composition series from $\{e\}$ to G. Thus pyramidal groups are solvable. In 1987 M. A. Berger, A. Felzenbaum and A. S. Fraenkel [BFF4] verified the Herzog-Schönheim conjecture for pyramidal groups.

In the 1950's Erdős proposed the following famous unsolved problem (see [Gu]): Whether for any arbitrarily large $c > 0$ there exists a cover (1.3) of $\mathbb{Z}$ satisfying $c < n_1 < \cdots < n_k$? A more general question is as follows:

Open Question. *Let G be a group and let M be a given positive integer. Whether for any $N > 0$ there is a finite cover (1.1) of G with each of the indices $n_i = [G : G_i]$ greater than N and occurring at most M times?*

For uniform covers of groups by cosets of subnormal subgroups, we are going to confirm the generalized Herzog-Schönheim conjecture and answer the above open question negatively! Actually we will make further progress.

Let's introduce our basic notations.

For $n \in \mathbb{Z}^+$ we let $P(n)$ be the set of prime divisors of n. For a prime p and a positive integer n, by $\mathrm{ord}_p n$ we mean the largest integer h such that $p^h \mid n$. For $n_1, \ldots, n_k \in \mathbb{Z}^+$, $(n_1, \ldots, n_k)$ (or $(n_i)_{1 \leqslant i \leqslant k}$) and $[n_1, \ldots, n_k]$ (or $[n_i]_{1 \leqslant i \leqslant k}$) stand for their greatest common divisor and least common multiple respectively. For a real number x the integral part of x is denoted by $\lfloor x \rfloor$. We also adopt conventional symbols $\sim$, o and O in analytic number theory (see, e.g. [Ap]). For convenience we regard $\sum_{i \in \emptyset} x_i$ and $\prod_{i \in \emptyset} x_i$ as 0 and 1 respectively.

For a subgroup H of a group G, let H_G denote the core (i.e. normal interior) of H in G, and let G/H stand for the quotient group $\{xH : x \in G\}$ if H is normal in G. For a union X of some left cosets of the subgroup H, by $[X : H]$ we mean the number of left cosets of H contained in X. Sylow p-subgroup and Hall ω-subgroup have their usual meanings where p is a prime and ω is a set of primes (cf. [Ro]). When group G and subgroups $G_1, \ldots, G_k$ are given, we let $\bigcap_{i \in I} G_i$ make sense for all $I \subseteq \{1, \ldots, k\}$ by regarding $\bigcap_{i \in \emptyset} G_i$ as G.

The main result of this paper is Theorem 4.3, for the sake of clarity we state here a simpler version.

Theorem 1.1. *Let (1.1) be a nontrivial uniform cover of a group G with*

$$n_1 = [G : G_1] \leqslant \cdots \leqslant n_k = [G : G_k]. \tag{1.4}$$

Suppose that all the G_i are subnormal in G, or G/H is a solvable group having a normal Sylow p-subgroup where H is the largest normal subgroup of G contained in all the G_i and p is the largest prime divisor of $|G/H|$. Then the indices $n_1, \ldots, n_k$ cannot be pairwise distinct. Moreover, if $|\{1 \leqslant i \leqslant k : n_i = n\}| \leqslant M$ for all $n \in \mathbb{Z}^+$ then we have

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log \log M) \tag{1.5}$$

where the logarithm has the natural base $e = 2.718...$, $\gamma = 0.577...$ is the Euler constant and the O-constant is absolute.

The next section contains some useful lemmas concerning indices of subgroups and normal Hall subgroups. In Sections 3 we are going to study unions of cosets. We will investigate uniform covers and obtain the main results in the last section.

2. Lemmas on Indices of Subgroups and Normal Hall Subgroups

Lemma 3.1(ii) of [Su8] can be restated as follows.

Lemma 2.1. *Let G be a group and $G_1, \ldots, G_k$ be subnormal subgroups of G with finite index. Then $[G : \bigcap_{i=1}^{k} G_i] \mid \prod_{i=1}^{k} [G : G_i]$ and hence*

$$P\left(\left[G : \bigcap_{i=1}^{k} G_i\right]\right) = \bigcup_{i=1}^{k} P([G : G_i]). \tag{2.1}$$

Remark 2.1. If $G_1, \ldots, G_k$ are subgroups of a group G with finite index, then $[G : \bigcap_{i=1}^{k} G_i] \leqslant \prod_{i=1}^{k} [G : G_i] < \infty$ by Poincaré's theorem. Lemma 2.1 can be viewed as an important number-theoretic property of subnormality, it is the main reason why covers involving subnormal subgroups are better behaved than general covers.

Lemma 2.2. *Let G be a group and H be a subnormal subgroup of G with finite index. Then*

$$P(|G/H_G|) = P([G : H]). \tag{2.2}$$

Proof. Let $\{a_i H\}_{i=1}^{k}$ be a partition of G into left cosets of H. Then $H_G = \bigcap_{g \in G} gHg^{-1} = \bigcap_{i=1}^{k} \bigcap_{h \in H} a_i h H h^{-1} a_i^{-1} = \bigcap_{i=1}^{k} a_i H a_i^{-1}$. Since those $a_i H a_i^{-1}$ are subnormal subgroups with index $k = [G : H]$, (2.2) follows from Lemma 2.1. $\square$

Corollary 2.1. *Let G be a finite group and H be a Hall subgroup of G. If H is subnormal in G, then H must be normal in G.*

Proof. By Lemma 2.2, $P(|G/H_G|) = P([G : H])$. So no prime factor of $|H|$ can divide $|G/H_G| = [G : H]|H/H_G|$. Thus H coincides with H_G. $\square$

Lemma 2.3. *Let G be a group and H, K be normal subgroups of G with finite index. Let ω be a set of primes. Then both G/H and G/K have normal Hall ω-subgroups, if and only if $G/(H \cap K)$ has a normal Hall ω-subgroup.*

Proof. Suppose that $G/(H \cap K)$ has a normal Hall ω-subgroup $F/(H \cap K)$ where $F \supseteq (H \cap K)$. Then F is normal in G and FH/H is normal in G/H. Observe that $|FH/H| = |F/(F \cap H)|$ divides $|F/(H \cap K)|$ and hence $P(|FH/H|) \subseteq \omega$. As $[G : FH]$ divides $[G : F]$, FH/H is a Hall ω-subgroup of G/H. Similarly, G/K has a normal Hall ω-subgroup.

Now assume that G/H and G/K have normal Hall ω-subgroups H^*/H and K^*/K respectively. Then $(H^* \cap K^*)/(H \cap K)$ is normal in $G/(H \cap K)$. Let $\bar{\omega}$ be the set of primes not in ω. In light of Lemma 2.1,

$$P([G : H^* \cap K^*]) = P([G : H^*]) \cup P([G : K^*]) \subseteq \bar{\omega}.$$

As $H \cap K = (H^* \cap K) \cap (H \cap K^*)$, we have

$$P(|(H^* \cap K^*)/(H \cap K)|) = P([H^* \cap K^* : H^* \cap K]) \cup P([H^* \cap K^* : H \cap K^*])$$
$$= P([(H^* \cap K^*)K : K]) \cup P([(H^* \cap K^*)H : H]) \subseteq P(|K^*/K|) \cup P(|H^*/H|) \subseteq \omega.$$

So $(H^* \cap K^*)/(H \cap K)$ is a Hall ω-subgroup of $G/(H \cap K)$. We are done. □

Let G be a finite group and p be a prime number. Then Sylow p-subgroups of G are just Hall $\{p\}$-subgroups of G. If G has a normal Sylow p-subgroup S, then by Sylow's theorem (cf. [Ro, 5.9]) S is the only Sylow p-subgroup of G.

Lemma 2.4. *Let G be a group and H a subgroup of G with finite index. Then, for any $p \in P(|G/H_G|) \setminus P([G : H])$, G/H_G doesn't have a normal Sylow p-subgroup.*

Proof. Let K/H_G be a Sylow p-subgroup of H/H_G where K is a subgroup of H containing H_G. Since $p \nmid [G : H]$, K/H_G is also a Sylow p-subgroup of G/H_G. If G/H_G has a normal Sylow p-subgroup, then K/H_G is the unique Sylow p-subgroup of G/H_G and therefore K is normal in G, thus $K \subseteq H_G$ and hence $|K/H_G| = 1$, this leads to a contradiction since $p \mid |G/H_G|$. □

Remark 2.2. We can extend Lemma 2.4 as follows: Let G be a group and ω be a set of primes. If H is a subgroup of G with finite index and G/H_G has a normal Hall ω-subgroup, then $P([G : H]) \cap \omega \neq \emptyset$ if and only if $P(|G/H_G|) \cap \omega \neq \emptyset$.

Lemma 2.5. *Let G be a finite group and p be a prime dividing $|G|$. Then G is a solvable group with a normal Sylow p-subgroup, if and only if there is a composition series $\{e\} = H_0 \subset H_1 \subset \cdots \subset H_n = G$ from $\{e\}$ to G for which all quotients $H_1/H_0, \ldots, H_n/H_{n-1}$ have prime order, and if a quotient is not of order p then neither is the next quotient.*

Proof. For the 'only if' direction, we suppose that G is solvable and that S is a normal Sylow p-subgroup of G. By [Ro, 5.31] there must be a composition series from $\{e\}$ to p-group S whose quotients are of order p. As G/S is solvable and each prime divisor of $|G/S|$ is different from p, there exists a composition series from S to G such that the order of any quotient is a prime other than p. Combining these we obtain a desired composition series from $\{e\}$ to G.

Now we consider the 'if' direction. Let $\{e\} = H_0 \subset H_1 \subset \cdots \subset H_i \subset H_{i+1} \subset \cdots \subset H_n = G$ be a composition series from $\{e\}$ to G for which $|H_1/H_0| = \cdots = |H_i/H_{i-1}| = p$ and $|H_{i+1}/H_i|, \cdots, |H_n/H_{n-1}|$ are primes different from p. Observe that H_i is a Sylow p-subgroup of G. By Corollary 2.1 subnormal subgroup H_i is normal in G. So G is a solvable group with normal Sylow p-subgroup H_i.

The proof of Lemma 2.5 is now complete. □

Corollary 2.2. *Let G be any pyramidal group. For the largest prime factor p of $|G|$, G has a normal Sylow p-subgroup.*

Proof. This follows immediately from Lemma 2.5. □

Remark 2.3. We can show that a group is pyramidal if and only if it has a Sylow tower. Also, if a group is pyramidal then so are its subgroups and quotient groups.

3. On Unions of Cosets

In [Su1] it was asked whether for subgroups $G_1, \ldots, G_k$ and elements $a_1, \ldots, a_k$ of a finite group G we always have

$$\left| \bigcup_{i=1}^{k} a_i G_i \right| \geqslant \left| \bigcup_{i=1}^{k} G_i \right|.$$

In 1991 Tomkinson gave a negative answer for $G = C_2 \times C_2$ where C_2 is the cyclic group of order 2. On the other hand, we have

Theorem 3.1. *Let G be a group and H its subgroup with $[G : H] < \infty$. Let $G_1, \ldots, G_k$ be subgroups of G containing H. Assume that either $G_1, \ldots, G_k$ are subnormal in G or there is a composition series from H to G whose quotients have prime order. Then for any $a_1, \ldots, a_k \in G$ we have*

$$|\{xH \colon x \in a_i G_i \ \text{ for some } i = 1, \ldots, k\}|$$
$$\geqslant |\{0 \leqslant n < [G : H] \colon [G : G_i] \mid n \ \text{ for some } i = 1, \ldots, k\}|,$$

i.e.,

$$\left[\bigcup_{i=1}^{k} a_i G_i : H \right] \geqslant \left[\bigcup_{i=1}^{k} [G : G_i]\mathbb{Z} : [G : H]\mathbb{Z} \right]. \tag{3.1}$$

To prove it we need some preparations.
For $R \subseteq \mathbb{Z}^+$ we define

$$D(R) = \{d \in \mathbb{Z}^+ \colon d \mid m \text{ for some } m \in R\};$$

if $k \in \mathbb{Z}^+$ then $R \subseteq D(R) \subseteq D(kR)$ where $kR = \{kr \colon r \in R\}$. Obviously $D(\emptyset) = \emptyset$ and $D(R_1 \cup R_2) = D(R_1) \cup D(R_2)$ for $R_1, R_2 \subseteq \mathbb{Z}^+$.

Following Berger et al. [BFF4], we introduce a measure μ on finite subsets of $\mathbb{Z}^+$ through $\mu(\{m\}) = \varphi(m)$ where φ is Euler's totient function. For $m = 1, 2, 3, \cdots$ Gauss' identity $\sum_{d \mid m} \varphi(d) = m$ shows that $\mu(D(\{m\})) = m$.

For $k, m \in \mathbb{Z}^+$ and any finite $R \subseteq \mathbb{Z}^+$ we have

$$\mu(D(k(R \cup \{m\}))) = \mu(D(kR) \cup D(\{km\}))$$
$$= \mu(D(kR)) + \mu(D(\{km\})) - \mu(D(kR) \cap D(\{km\}))$$
$$= \mu(D(kR)) + km - \mu(D(\{(kr, km) \colon r \in R\}))$$
$$= \mu(D(kR)) + k\mu(D(\{m\})) - \mu(D(kR'))$$

where $R' = \{(r, m) \colon r \in R\}$. From this by induction we can establish

Lemma 3.1. *Let k be a positive integer and R be a finite subset of $\mathbb{Z}^+$. Then*

$$\mu(D(kR)) = k\mu(D(R)). \tag{3.2}$$

Remark 3.1. The lemma was first observed by Berger et al. [BFF4].

Lemma 3.2. *Let Γ be a family of finite sets such that whenever $S, T \in \Gamma$ one has $S \cap T \in \Gamma$ and $|S \cap T| = (|S|, |T|)$. For any finite subfamily Δ of Γ we have*

$$\left| \bigcup_{S \in \Delta} S \right| = \mu(D(\{|S|: S \in \Delta\})). \tag{3.3}$$

Proof. Since $\mu(D(\emptyset)) = \mu(\emptyset) = 0$, (3.3) holds trivially if Δ is empty .

Now let $\Delta_0 \subseteq \Gamma$ have cardinality $n \in \mathbb{Z}^+$ and assume (3.3) for any $\Delta \subseteq \Gamma$ with smaller cardinality. Suppose $T \in \Delta_0$ and let $\Delta_0' = \Delta_0 \setminus \{T\}$. By the induction hypothesis, we have

$$\left| \bigcup_{S \in \Delta_0} S \right| = \left| T \cup \bigcup_{S \in \Delta_0'} S \right| = \left| \bigcup_{S \in \Delta_0'} S \right| + |T| - \left| \bigcup_{S \in \Delta_0'} S \cap T \right|$$

$$= \mu(D(\{|S|: S \in \Delta_0'\})) + \mu(D(\{|T|\})) - \mu(D(\{|S \cap T|: S \in \Delta_0'\}))$$

$$= \mu(D(\{|S|: S \in \Delta_0'\})) + \mu(D(\{|T|\})) - \mu(D(\{|S|: S \in \Delta_0'\}) \cap D(\{|T|\}))$$

$$= \mu(D(\{|S|: S \in \Delta_0'\}) \cup D(\{|T|\})) = \mu(D(\{|S|: S \in \Delta_0\})).$$

This concludes the proof by induction. $\square$

Lemma 3.3. *Let G be a group and H be a subgroup of G with finite index. Suppose that $G_1, \cdots, G_k$ are subgroups of G containing H. Then*

$$\begin{aligned} &\mu(D(\{[G_i : H]: 1 \leqslant i \leqslant k\})) \\ &= |\{0 \leqslant n < [G : H]: [G : G_i] \mid n \text{ for some } 1 \leqslant i \leqslant k\}|. \end{aligned} \tag{3.4}$$

Proof. Clearly $N = [G : H]$ is a multiple of those $n_i = [G : G_i]$ with $1 \leqslant i \leqslant k$. For each divisor d of N we let $X_d = \{0 \leqslant x < N: x \in d\mathbb{Z}\}$. If $m, n \in \mathbb{Z}^+$ divide N, then $X_m \cap X_n = X_{[m,n]}$ has cardinality $N/[m, n] = (|X_m|, |X_n|)$. Applying Lemma 3.2 to the family

$$\Gamma = \{X_d: d \in \mathbb{Z}^+ \text{ and } d \mid N\},$$

we obtain that

$$\left| \bigcup_{i=1}^{k} X_{n_i} \right| = \mu\left(D(\{|X_{n_i}|: 1 \leqslant i \leqslant k\})\right) = \mu\left(D\left(\left\{\frac{N}{n_1}, \dots, \frac{N}{n_k}\right\}\right)\right).$$

So (3.4) holds. □

Proof of Theorem 3.1. We use induction on $[G:H]$. If $G_1 = \cdots = G_k = G$, then

$$\left|\left\{aH: a \in \bigcup_{i=1}^{k} a_i G_i\right\}\right| = [G:H] = \left|\left\{0 \leqslant n < [G:H]: n \in \bigcup_{i=1}^{k} [G:G_i]\mathbb{Z}\right\}\right|.$$

Thus the case $[G:H] = 1$ is trivial. So we proceed to the induction step with $[G:H] > 1$ and assume that $G_j \neq G$ for some $1 \leqslant j \leqslant k$.

Case 1. $G_1, \ldots, G_k$ are subnormal in G. As $G_j \neq G$ there exists a proper maximal normal subgroup H^* of G containing G_j. Observe that each $G_i \cap H^*$ is subnormal in H^* since G_i is subnormal in G.

Case 2. There exists a composition series from H to G whose quotients have prime order. Since $H \neq G$ there is a normal subgroup H^* of prime index in G for which there exists a composition series from H to H^* whose quotients are of prime order.

In either case, $H \subseteq G_i \cap H^* \subseteq H^*$ and $[H^*:H] < [G:H]$. Also, $G_i H^*$ coincides with G or H^*.

Write $G/H^* = \{g_1 H^*, \cdots, g_h H^*\}$ where $h = [G:H^*]$. Set

$$I_s = \{1 \leqslant i \leqslant k: a_i G_i \cap g_s H^* \neq \emptyset\} \quad \text{for } s = 1, \ldots, h.$$

For each $i = 1, \ldots, k$ clearly $a_i G_i \cap g_s H^* \neq \emptyset$ for some $1 \leqslant s \leqslant h$, so

$$I_1 \cup \cdots \cup I_h = \{1, \ldots, k\}. \tag{3.5}$$

For $I = I_1 \cap \cdots \cap I_h$ we have

$$\begin{aligned}
I &= \{1 \leqslant i \leqslant k: G_i \cap a_i^{-1} g_s H^* \neq \emptyset \text{ for all } s = 1, \ldots, h\} \\
&= \{1 \leqslant i \leqslant k: xG_i \cap yH^* = x(G_i \cap x^{-1}yH^*) \neq \emptyset \text{ for all } x, y \in G\} \\
&= \{1 \leqslant i \leqslant k: G_i H^* = G\} \quad \text{(by Lemma 2.1 of [Su8])}.
\end{aligned}$$

Let $R = \{[G_i \cap H^*: H]: i \in I\}$. Then $hR = \{[G_i:H]: i \in I\}$ since $[G_i:G_i \cap H^*] = [G_i H^*: H^*] = h$ for all $i \in I$.

Let $s \in \{1, \ldots, h\}$ and $R_s = \{[G_i \cap H^*: H]: i \in I_s \setminus I\}$. For $i \in I_s \setminus I$, as $G_i H^* = H^*$ we have $G_i \subseteq H^*$ and $a_i G_i \subseteq g_s H^*$. So $R_s = \{[G_i:H]: i \in I_s \setminus I\}$.

If $i \in I_s$, then $g_s^{-1} a_i G_i \cap H^*$ is nonempty and hence it is a left coset of $G_i \cap H^*$ in H^*. Clearly

$$\bigcup_{i=1}^{k} a_i G_i = \bigcup_{s=1}^{h} \bigcup_{i=1}^{k} a_i G_i \cap g_s H^* = \bigcup_{s=1}^{h} \bigcup_{i \in I_s} a_i G_i \cap g_s H^*$$

and so

$$\left[\bigcup_{i=1}^{k} a_i G_i : H\right] = \sum_{s=1}^{h} \left[\bigcup_{i \in I_s} a_i G_i \cap g_s H^* : H\right] = \sum_{s=1}^{h} \left[\bigcup_{i \in I_s} g_s^{-1} a_i G_i \cap H^* : H\right].$$

Thus, by the induction hypothesis,

$$\left[\bigcup_{i=1}^{k} a_i G_i : H\right] \geqslant \sum_{s=1}^{h} \left|\left\{0 \leqslant n < [H^* : H] : n \in \bigcup_{i \in I_s} [H^* : G_i \cap H^*]\mathbb{Z}\right\}\right|$$

$$= \sum_{s=1}^{h} \mu(D(\{[G_i \cap H^* : H] : i \in I_s\})) \quad \text{(by Lemma 3.3)}$$

$$= \sum_{s=1}^{h} \mu(D(R \cup R_s)) = h\mu(D(R)) + \sum_{s=1}^{h} \mu(D(R_s) \setminus D(R))$$

$$\geqslant \mu(D(hR)) + \sum_{s=1}^{h} \mu(D(R_s) \setminus D(hR)) \quad \text{(by Lemma 3.1)}.$$

It follows that

$$\left[\bigcup_{i=1}^{k} a_i G_i : H\right] \geqslant \mu\left(D(hR) \cup \bigcup_{s=1}^{h} (D(R_s) \setminus D(hR))\right) = \mu\left(\bigcup_{s=1}^{h} D(hR \cup R_s)\right)$$

$$= \mu\left(\bigcup_{s=1}^{h} D(\{[G_i : H] : i \in I_s\})\right) = \mu(D(\{[G_i : H] : 1 \leqslant i \leqslant k\}))$$

$$= \left|\left\{0 \leqslant n < [G : H] : n \in \bigcup_{i=1}^{k} [G : G_i]\mathbb{Z}\right\}\right| \quad \text{(by Lemma 3.3)}.$$

This completes the proof. $\square$

Remark 3.2. A theorem of C. A. Rogers (cf. [HR]) indicates that if $a_i \in \mathbb{Z}$ and $n_i \in \mathbb{Z}^+$ for $i = 1, \ldots, k$ then for any positive multiple N of $n_1, \ldots, n_k$ we have

$$\left|\left\{0 \leqslant x < N : x \in \bigcup_{i=1}^{k} a_i + n_i\mathbb{Z}\right\}\right| \geqslant \left|\left\{0 \leqslant x < N : x \in \bigcup_{i=1}^{k} n_i\mathbb{Z}\right\}\right|, \qquad (3.6)$$

this is just our Theorem 3.1 in the case where G is the infinite cyclic group $\mathbb{Z}$. (It should be mentioned that Simpson [Si] presented this as his Lemma 2.3 but gave a wrong proof.) In view of Lemma 3.3, Lemma IV of [BFF4] is equivalent to our Theorem 3.1 in the case where G is a pyramidal group and H is the smallest subgroup $\{e\}$.

As in additive number theory, for any $S \subseteq \mathbb{Z}$ we let $d(S)$ denote the asymptotic density

$$\lim_{N \to +\infty} \frac{|\{0 \leqslant n < N : n \in S\}|}{N}$$

if the limit exists. It is easy to see that for system (1.3) we have

$$d\left(\bigcup_{i=1}^{k} a_i + n_i\mathbb{Z}\right) = \frac{1}{N}\left|\left\{0 \leqslant x < N : x \in \bigcup_{i=1}^{k} a_i + n_i\mathbb{Z}\right\}\right|$$

where N is any positive multiple of $[n_1, \ldots, n_k]$.

Here we restate Lemma 2 of [Su2] (proved by the inclusion-exclusion principle).

Lemma 3.4. *Let $n_1, \ldots, n_k$ be positive integers and let P be a finite set of primes such that $P(n_i) \subseteq P$ for all $i = 1, \ldots, k$. Then*

$$d\left(\bigcup_{i=1}^{k} n_i \mathbb{Z} \right) = \left(\prod_{p \in P} \frac{p-1}{p} \right) \sum_{\substack{n \in \bigcup_{i=1}^{k} n_i \mathbb{Z}^+ \\ P(n) \subseteq P}} \frac{1}{n}. \tag{3.7}$$

Now we are able to give

Theorem 3.2. *Let G be a group and $G_1, \ldots, G_k, H$ be subgroups of G with finite index. Let $a_1, \ldots, a_k \in G$ and assume that the union of $a_1 G_1, \ldots, a_k G_k$ coincides with a union of some left cosets of H. Let $h = [G : H]$ and $n_i = [G : G_i]$ for $i = 1, \ldots, k$. Then we have*

$$\frac{(n_1, \ldots, n_k)}{(h, n_1, \ldots, n_k)} \leqslant \sup_{n \in \mathbb{Z}^+} |\{1 \leqslant i \leqslant k \colon n_i = n\}| \sum_{d | \frac{[n_1, \ldots, n_k]}{(n_1, \ldots, n_k)}} \frac{1}{d} \tag{3.8}$$

in the following four cases.

(a) *All the G_i are subnormal and H is normal in G.*

(b) *All the G_i are normal and H is subnormal in G.*

(c) *All the G_i are normal in G and $G / \bigcap_{i=1}^{k} G_i$ is solvable.*

(d) *H is normal in G, and G/H or each $G/(G_i)_G$ is solvable.*

Proof. In either case $G_i H = H G_i$ for $i = 1, \ldots, k$. Clearly $G_i H$ is subnormal in G for every $i = 1, \ldots, k$ in case (a) or case (b), and there is a composition series from H or $\bigcap_{i=1}^{k}(G_i)_G$ to G whose quotients have prime order in case (c) or case (d). Note that

$$\bigcup_{i=1}^{k} a_i G_i = \left(\bigcup_{i=1}^{k} a_i G_i \right) H = \bigcup_{i=1}^{k} a_i G_i H.$$

With the help of Theorem 3.1, for a suitable $F \in \{\bigcap_{i=1}^{k}(G_i)_G, H\}$ we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = \sum_{i=1}^{k} \frac{[a_i G_i : F]}{[G : F]}$$

$$\geqslant \frac{1}{[G : F]} \left[\bigcup_{i=1}^{k} a_i G_i : F \right] = \frac{1}{[G : F]} \left[\bigcup_{i=1}^{k} a_i G_i H : F \right]$$

$$\geqslant \frac{1}{[G : F]} \left| \left\{ 0 \leqslant n < [G : F] \colon [G : G_i H] \mid n \text{ for some } i = 1, \ldots, k \right\} \right|$$

$$= d\left(\bigcup_{i=1}^{k} [G : G_i H] \mathbb{Z} \right).$$

Let $S = \{n_1, \ldots, n_k\}$, $P = \bigcup_{n \in S} P(n)$ and $\bar{P} = \{n \in \mathbb{Z}^+ : P(n) \subseteq P\}$. Then

$$\bigcup_{i=1}^{k} P([G : G_i H]) \subseteq \bigcup_{i=1}^{k} P(n_i) = P, \quad \text{i.e.} \quad \{[G : G_i H] : 1 \leqslant i \leqslant k\} \subseteq \bar{P}.$$

Obviously $(n_1, \ldots, n_k)/(h, n_1, \ldots, n_k) = [h, (n_1, \ldots, n_k)]/h$ divides $n_i/(h, n_i) = [h, n_i]/h$ and $[G : G_i H]$ divides (h, n_i), therefore

$$\frac{n_i}{(n_1, \ldots, n_k)/(h, n_1, \ldots, n_k)} \in \bar{P} \cap \bigcup_{j=1}^{k} [G : G_j H]\mathbb{Z}.$$

Clearly $[n_1, \ldots, n_k]/(n_1, \ldots, n_k)$ can be written in the form $\prod_{p \in P} p^{\delta_p}$ where $\delta_p \in \mathbb{N} = \{0, 1, 2, \cdots\}$. For any $p \in P$ and $1 \leqslant i, j \leqslant k$ we have

$$\operatorname{ord}_p n_i - \operatorname{ord}_p n_j \leqslant \operatorname{ord}_p [n_1, \ldots, n_k] - \operatorname{ord}_p (n_1, \ldots, n_k) = \delta_p.$$

So, if $n, n' \in S$, $k_p, l_p \in \mathbb{N}$ and

$$n \prod_{p \in P} p^{k_p(1+\delta_p)} = n' \prod_{p \in P} p^{l_p(1+\delta_p)},$$

then $k_p = l_p$ for all $p \in P$ and hence $n = n'$.

Let

$$M = \sup_{n \in \mathbb{Z}^+} |\{1 \leqslant i \leqslant k : n_i = n\}| = \max_{n \in S} \sum_{\substack{i=1 \\ n_i = n}}^{k} 1.$$

In view of Lemma 3.4 and the above,

$$\sum_{i=1}^{k} \frac{1}{n_i} \geqslant d\left(\bigcup_{i=1}^{k} [G : G_i H]\mathbb{Z} \right) = \left(\prod_{p \in P} \frac{p-1}{p} \right) \sum_{m \in \bar{P} \cap \bigcup_{i=1}^{k} [G:G_i H]\mathbb{Z}} \frac{1}{m}$$

$$\geqslant \left(\prod_{p \in P} \frac{p-1}{p} \right) \sum_{n \in S} \left(\frac{n}{(n_1, \ldots, n_k)/(h, n_1, \ldots, n_k)} \right)^{-1} \prod_{p \in P} \sum_{i=0}^{\infty} \frac{1}{p^{i(1+\delta_p)}}$$

$$= \sum_{n \in S} \frac{1}{n} \cdot \frac{(n_1, \ldots, n_k)}{(h, n_1, \ldots, n_k)} \prod_{p \in P} \left(\frac{p-1}{p} \middle/ \left(1 - \frac{1}{p^{1+\delta_p}} \right) \right)$$

$$= \frac{1}{M} \sum_{n \in S} \frac{M}{n} \cdot \frac{(n_1, \ldots, n_k)}{(h, n_1, \ldots, n_k)} \prod_{p \in P} \frac{p^{\delta_p}}{1 + p + \cdots + p^{\delta_p}}$$

$$\geqslant \frac{1}{M} \sum_{i=1}^{k} \frac{1}{n_i} \cdot \frac{(n_1, \ldots, n_k)}{(h, n_1, \ldots, n_k)} \prod_{p \in P} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\delta_p}} \right)^{-1}.$$

Therefore

$$\frac{(n_1, \ldots, n_k)}{(h, n_1, \ldots, n_k)} \leqslant M \prod_{p \in P} \left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{\delta_p}} \right) = M \sum_{d \mid \frac{[n_1, \ldots, n_k]}{(n_1, \ldots, n_k)}} \frac{1}{d}.$$

We are done. $\square$

Our Theorem 3.2 is powerful, it will be applied in Section 4.

4. On Uniform Covers

Lemma 4.1. *Let* (1.1) *be a finite system of left cosets in a group* G. *Then*

$$K_{\mathcal{A}} = \{x \in G\colon w_{\mathcal{A}}(gx) = w_{\mathcal{A}}(g) \text{ for all } g \in G\} \tag{4.1}$$

is a subgroup of G *containing* $\bigcap_{i=1}^{k} G_i$. *For any nonempty subset* I *of* $\{1, \ldots, k\}$, *the union* $\bigcup_{i \in I} a_i G_i$ *coincides with a union of some left cosets of* $K_{\mathcal{A}} \cap \bigcap_{j \in \bar{I}} G_j$ *where* $\bar{I} = \{1, \ldots, k\} \setminus I$.

Proof. If $x, y \in K_{\mathcal{A}}$ then $w_{\mathcal{A}}(gxy^{-1}) = w_{\mathcal{A}}(gxy^{-1}y) = w_{\mathcal{A}}(gx) = w_{\mathcal{A}}(g)$ for all $g \in G$. So $K_{\mathcal{A}}$ is a subgroup of G. For $g \in G$ and $x \in G_i$, clearly $gx \in a_i G_i$ if and only if $g \in a_i G_i x^{-1} = a_i G_i$. Thus $K_{\mathcal{A}} \supseteq \bigcap_{i=1}^{k} G_i$.

Let $g \in G$ and $x \in K_{\mathcal{A}} \cap \bigcap_{j \in \bar{I}} G_j$. For $j \in \bar{I}$ we have $gx \in a_j G_j \Leftrightarrow g \in a_j G_j$. Therefore

$$|\{i \in I\colon gx \in a_i G_i\}| = w_{\mathcal{A}}(gx) - |\{j \in \bar{I}\colon gx \in a_j G_j\}|$$
$$= w_{\mathcal{A}}(g) - |\{j \in \bar{I}\colon g \in a_j G_j\}| = |\{i \in I\colon g \in a_i G_i\}|.$$

It follows that

$$g \in \bigcup_{i \in I} a_i G_i \Longrightarrow g\left(K_{\mathcal{A}} \cap \bigcap_{j \in \bar{I}} G_j\right) \subseteq \bigcup_{i \in I} a_i G_i.$$

So $X = \bigcup_{i \in I} a_i G_i$ is identical with $\bigcup_{g \in X} g(K_{\mathcal{A}} \cap \bigcap_{j \in \bar{I}} G_j)$. We are done. $\quad\square$

Theorem 4.1. *Let* (1.1) *be a nontrivial uniform cover of a group* G *by left cosets. Let* $n_i = [G : G_i]$ *for* $i = 1, \ldots, k$ *and* $[n_1, \ldots, n_k] = \prod_{t=1}^{r} p_t^{\alpha_t}$ *where* $p_1, \ldots, p_r$ *are distinct primes and* $\alpha_1, \ldots, \alpha_r$ *are positive integers. Let*

$$\beta_r = \min\{1 \leqslant \beta \leqslant \alpha_r\colon \beta = \mathrm{ord}_{p_r} n_i \text{ for some } i = 1, \ldots, k\}, \tag{4.2}$$

$$\varepsilon_r = \left(1 - \frac{1}{p_r^{\alpha_r - \beta_r + 1}}\right) \prod_{0 < t < r} \left(1 - \frac{1}{p_t^{\alpha_t + 1}}\right) \tag{4.3}$$

and

$$M_r = \max\left\{|\{1 \leqslant i \leqslant k\colon n_i = n_j\}|\colon 1 \leqslant j \leqslant k \;\&\; p_r \mid n_j\right\}. \tag{4.4}$$

Then we have

$$p_r^{\beta_r} \leqslant \varepsilon_r M_r \prod_{t=1}^{r} \frac{p_t}{p_t - 1} \tag{4.5}$$

providing the following (a) *and* (b), *or* (c) *in the case* $p_1 < \cdots < p_r$.

(a) *If not all the* G_i *with* $p_r \mid n_i$ *are subnormal in* G, *then all the* $G/(G_i)_G$ *with* $p_r \mid n_i$, *or those with* $p_r \nmid n_i$, *are solvable.*

(b) *For each* i *with* $n_i > p_r$ *and* $p_r \nmid n_i$, *if* G_i *is not subnormal in* G *then* $G/(G_i)_G$ *has a normal Sylow* p_r-*subgroup.*

(c) $\bar{G} = G/(\bigcap_{i=1}^{k} G_i)_G$ is a solvable group having a normal Sylow p-subgroup where p is the largest prime divisor of $|\bar{G}|$.

Proof. Suppose that $p_1 < \cdots < p_r$ and (c) holds. Since $(\bigcap_{i=1}^{k} G_i)_G = \bigcap_{i=1}^{k}(G_i)_G$, by Lemma 2.3 each $G/(G_i)_G$ is a solvable group having a normal Sylow p-subgroup. In view of Lemma 2.4, if $p \mid |G/(G_i)_G|$ then $p \mid n_i$. On the other hand

$$P(|\bar{G}|) = \bigcup_{i=1}^{k} P(|G/(G_i)_G|) \supseteq \bigcup_{i=1}^{k} P(n_i) = \{p_1, \ldots, p_r\},$$

so we have $p = p_r$. Therefore both (a) and (b) hold.

Below we prove (4.5) under the conditions (a) and (b).

Let $I = \{1 \leqslant i \leqslant k \colon p_r \mid n_i\}$ and $\bar{I} = \{1, \ldots, k\} \setminus I$. Since $w_{\mathcal{A}}$ is constant, $K_{\mathcal{A}} = G$. By Lemma 4.1, $\bigcup_{i \in I} a_i G_i$ coincides with a union of some left cosets of $\bigcap_{j \in \bar{I}} G_j$. Let $H = (\bigcap_{j \in \bar{I}} G_j)_G = \bigcap_{j \in \bar{I}}(G_j)_G$. Then G/H is finite and $\bigcup_{i \in I} a_i G_i$ is a union of finitely many cosets of H. Note that G/H is solvable if and only if $G/(G_j)_G$ is solvable for all $j \in \bar{I}$ (cf. [Ro, 7.46 and 7.50]). By condition (a) and Theorem 3.2, we have

$$\frac{(n_i)_{i \in I}}{(|G/H|, (n_i)_{i \in I})} \leqslant \sup_{n \in \mathbb{Z}^+} |\{i \in I \colon n_i = n\}| \sum_{d \mid \frac{[n_i]_{i \in I}}{(n_i)_{i \in I}}} \frac{1}{d}.$$

Therefore

$$\frac{[|G/H|, p_r^{\beta_r}]}{|G/H|} \leqslant \frac{[|G/H|, (n_i)_{i \in I}]}{|G/H|} \leqslant M_r \sum_{d \mid p_r^{\alpha_r - \beta_r} \prod_{0 < t < r} p_t^{\alpha_t}} \frac{1}{d}$$

$$= M_r \left(1 + \frac{1}{p_r} + \cdots + \frac{1}{p_r^{\alpha_r - \beta_r}}\right) \prod_{0 < t < r} \left(1 + \frac{1}{p_t} + \cdots + \frac{1}{p_t^{\alpha_t}}\right)$$

$$= M_r \frac{p_r^{\alpha_r - \beta_r + 1} - 1}{p_r^{\alpha_r - \beta_r}(p_r - 1)} \prod_{0 < t < r} \frac{p_t^{\alpha_t + 1} - 1}{p_t^{\alpha_t}(p_t - 1)} = \varepsilon_r M_r \prod_{t=1}^{r} \frac{p_t}{p_t - 1}.$$

Now it suffices to show that $p_r \nmid |G/H|$ under condition (b). In view of Lemma 2.1, $P(|G/H|) = \bigcup_{j \in \bar{I}} P(|G/(G_j)_G|)$. Let $j \in \bar{I}$. By [Ro, 4.14], $G/(G_j)_G$ can be embedded in the symmetric group S_{n_j}. If $n_j < p_r$, then $|S_{n_j}| = n_j! \not\equiv 0 \pmod{p_r}$ and hence $p_r \nmid |G/(G_j)_G|$. If G_j is subnormal in G, then by Lemma 2.2, $P(|G/(G_j)_G|) = P(n_j)$ doesn't contain p_r. When $G/(G_j)_G$ has a normal Sylow p_r-subgroup, $p_r \nmid |G/(G_j)_G|$ by Lemma 2.4. So $p_r \notin P([G : H])$ and we are done. $\square$

Remark 4.1. In [BS] N. Burshtein and Schönheim investigated disjoint covers of $\mathbb{Z}$ having moduli occurring at most twice. In 1976 Burshtein [Bu] conjectured that

for any disjoint cover (1.3) with each modulus occurring at most $M \in \mathbb{Z}^+$ times, if $p_1 < \cdots < p_r$ are the distinct prime divisors of $[n_1, \ldots, n_k]$ then

$$p_r \leqslant M \prod_{t=1}^{r} \frac{p_t}{p_t - 1}; \tag{4.6}$$

he also realized that the smallest modulus in such a disjoint cover cannot be arbitrarily large by his conjecture. The conjecture was later proved by Simpson [Si], and by Berger et al. [BFF3] independently. In [BFF2] and [BFF4] Berger et al. obtained the analogy of the Burshtein conjecture for partitions of finite nilpotent groups and pyramidal groups, their results follow from our Theorem 4.1 in view of Corollary 2.2.

Corollary 4.1. *Let G be a group of squarefree order, and (1.1) be a nontrivial uniform cover of G with $p_1 < \cdots < p_r$ being the prime divisors of the indices $[G : G_1], \ldots, [G : G_k]$. Then for some $n \equiv 0 \pmod{p_r}$ we have*

$$|\{1 \leqslant i \leqslant k \colon [G : G_i] = n\}| \geqslant \frac{p_1 \cdots p_r}{\prod_{0 < t < r}(p_t + 1)} \geqslant \max\left\{p_1, \frac{2p_r}{r + 1}\right\}. \tag{4.7}$$

Proof. As $\bar{G} = G/(\bigcap_{i=1}^{k} G_i)_G$ has squarefree order, by [Ro, Exercise 609] $\bar{G}$ is a solvable group having a normal Sylow p-subgroup where p is the largest prime divisor of $|\bar{G}|$. By Theorem 4.1, for some $j = 1, \ldots, k$ with $[G : G_j] \in p_r\mathbb{Z}$, we have

$$p_r \leqslant M\left(1 - \frac{1}{p_1^2}\right) \cdots \left(1 - \frac{1}{p_{r-1}^2}\right)\left(1 - \frac{1}{p_r}\right) \prod_{t=1}^{r} \frac{p_t}{p_t - 1}$$

where $M = |\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}|$. Thus

$$\frac{M}{p_r} \geqslant \left(1 + \frac{1}{p_1}\right)^{-1} \cdots \left(1 + \frac{1}{p_{r-1}}\right)^{-1} = \prod_{0 < t < r} \frac{p_t}{p_t + 1} = \prod_{0 < t < r}\left(1 - \frac{1}{p_t + 1}\right)$$

$$\geqslant \prod_{0 < t < r}\left(1 - \frac{1}{p_1 + t}\right) = \frac{p_1}{p_1 + r - 1} \geqslant \max\left\{\frac{p_1}{p_r}, \frac{2}{r + 1}\right\}$$

and the desired result follows. $\square$

Our progress on the Herzog-Schönheim conjecture is as follows.

Corollary 4.2. *Let G be a group and (1.1) be a nontrivial uniform cover of G by left cosets. Let r be the number of distinct prime divisors of $N = [[G : G_1], \ldots, [G : G_k]]$, and let p be any prime divisor of $|G/(\bigcap_{i=1}^{k} G_i)_G|$ greater than r (e.g. the largest prime divisor of N). Suppose that all those G_i with $[G : G_i] \geqslant p$ are subnormal in G and p divides N, or $G/(\bigcap_{i=1}^{k} G_i)_G$ is a solvable group having a*

normal Sylow p-subgroup. Then there is a pair $\{i, j\}$ with $1 \leqslant i < j \leqslant k$ such that $[G : G_i] = [G : G_j] \equiv 0 \pmod{p}$.

Proof. If $\bar{G} = G / \bigcap_{i=1}^{k} (G_i)_G$ is a solvable group having a normal Sylow p-subgroup, then so is each $G/(G_i)_G$ by Lemma 2.3, also p divides N by Lemmas 2.1 and 2.4.

Set $p_r = p$ and let $p_1, \ldots, p_{r-1}$ be the other $r-1$ distinct prime divisors of N. By Theorem 4.1 we have

$$p_r < |\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}| \prod_{t=1}^{r} \frac{p_t}{p_t - 1}$$

for some $j \in \{1, \ldots, k\}$ with $p \mid [G : G_j]$. Thus

$$|\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}| > p \prod_{t=1}^{r} \left(1 - \frac{1}{p_t}\right) \geqslant (r+1) \prod_{t=1}^{r} \left(1 - \frac{1}{t+1}\right) = 1.$$

Therefore $[G : G_i] = [G : G_j] \equiv 0 \pmod{p}$ for some $i = 1, \ldots, k$ with $i \neq j$. $\square$

For cyclic groups we can say something more general than Theorem 4.1.

Theorem 4.2. *Let* (1.1) *be a nontrivial uniform cover of a cyclic group G by cosets of subgroups G_i of indices n_i. Assume that $[n_1, \ldots, n_k] = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Let α be a positive integer in $\Lambda = \{\mathrm{ord}_{p_r} n_i \colon 1 \leqslant i \leqslant k\}$ and β be the largest integer in $\Lambda \cup \{0\}$ less than α. Then*

$$p_r^{\alpha - \beta} \leqslant \varepsilon \max_{\substack{1 \leqslant j \leqslant k \\ p_r^{\alpha} \mid n_j}} |\{1 \leqslant i \leqslant k \colon n_i = n_j\}| \prod_{t=1}^{r} \frac{p_t}{p_t - 1} \tag{4.8}$$

where

$$\varepsilon = \left(1 - \frac{1}{p_1^{\alpha_1 + 1}}\right) \cdots \left(1 - \frac{1}{p_{r-1}^{\alpha_{r-1} + 1}}\right) \left(1 - \frac{1}{p_r^{\alpha_r - \alpha + 1}}\right). \tag{4.9}$$

Consequently,

$$\max_{\substack{1 \leqslant j \leqslant k \\ \mathrm{ord}_{p_r} n_j = \alpha_r}} |\{1 \leqslant i \leqslant k \colon n_i = n_j\}| \geqslant p_r \prod_{0 < t < r} \frac{p_t - 1}{p_t} \geqslant \frac{p_r}{r}. \tag{4.10}$$

Proof. Let $I = \{1 \leqslant i \leqslant k \colon p_r^{\alpha} \mid n_i\}$ and $\bar{I} = \{1, \ldots, k\} \setminus I$. Set $H = \bigcap_{j \in \bar{I}} G_j$ and

$$M = \sup_{n \in \mathbb{Z}^+} |\{i \in I \colon n_i = n\}| = \max_{\substack{1 \leqslant j \leqslant k \\ p_r^{\alpha} \mid n_j}} |\{1 \leqslant i \leqslant k \colon n_i = n_j\}|.$$

As in the proof of Theorem 4.1 we have

$$\frac{[|G/H|, p_r^{\alpha}]}{|G/H|} \leqslant \varepsilon M \prod_{t=1}^{r} \frac{p_t}{p_t - 1}.$$

If $\bar{I} = \emptyset$ then $\mathrm{ord}_{p_r}|G/H| = 0 = \beta$. When $\bar{I} \neq \emptyset$ and $G = \langle a \rangle$, $H = \bigcap_{j \in \bar{I}} \langle a^{n_j} \rangle = \langle a^{[n_j]_{j \in \bar{I}}} \rangle$ and therefore $\mathrm{ord}_{p_r}|G/H| = \mathrm{ord}_{p_r}[n_j]_{j \in \bar{I}} = \beta$. So $[|G/H|, p_r^\alpha]/|G/H| = p_r^{\alpha - \beta}$ and hence (4.8) holds. If we take $\alpha = \alpha_r$ then $\varepsilon \leqslant 1 - p_r^{-1} = (p_r - 1)/p_r$ and the first inequality in (4.10) follows. For the second inequality in (4.10), we note that $\prod_{0 < t < r}(p_t - 1)/p_t \geqslant \prod_{0 < s < r} s/(s+1) = 1/r$. This ends our proof. $\square$

Remark 4.2. Let (1.3) be a disjoint cover of $\mathbb{Z}$ with each modulus occurring at most M times. Suppose that $p_1, \ldots, p_r$ are the distinct prime divisors of $n_1, \ldots, n_k$. In 1986 Simpson [Si] showed the inequality $p_r \leqslant M \prod_{0 < t < r} p_t/(p_t - 1)$. (In the case $r \geqslant 2$ and $p_1 < \cdots < p_r$, the weaker inequality $M \geqslant p_2(p_1 - 1)/p_1$ was first noted in [BFF1].) This improvement to the original Burshtein conjecture was strengthened in [Su2] where the author got Theorem 4.2 for disjoint covers of $\mathbb{Z}$. For any cyclic group G, Theorem 4.1 corresponds to Theorem 4.2 in the case $\alpha = \min(\Lambda \cap \mathbb{Z}^+)$.

From now on variable p will only take prime values as in number theory.

Lemma 4.2. *For $M \geqslant 2$, if $q > 1$ is an integer with $q < M \prod_{p \leqslant q} p/(p-1)$ then*

$$q < e^\gamma M \log M + O(M \log \log M) \ \text{ and } \ \pi(q) \leqslant e^\gamma M + O(M/\log M) \qquad (4.11)$$

where $\pi(q)$ is the number of primes not exceeding q and the O-constants are absolute.

Proof. A well-known theorem of Mertens (see Theorem 13.13 of [Ap]) asserts that

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right) \quad \text{ for } x \geqslant 2.$$

Thus for $x \in [2, +\infty)$ we have

$$\prod_{p \leqslant x} \frac{p}{p-1} = \frac{e^\gamma \log x}{1 + O(\frac{1}{\log x})} = (e^\gamma \log x)\left(1 + O\left(\frac{1}{\log x}\right)\right) = e^\gamma \log x + O(1).$$

(Note that $(1-z)^{-1} = 1 + z/(1-z) = 1 + O(z)$ when $|z| < 1/2$.) For $M \geqslant 2$ we let $c(M)$ be the smallest positive integer x such that $\prod_{p \leqslant x} p/(p-1) \leqslant x/M$, obviously $c(M) > 2$.

When $M' \geqslant M$, we have $c(M') \geqslant c(M)$ because

$$\frac{1}{c(M')} \prod_{p \leqslant c(M')} \frac{p}{p-1} \leqslant \frac{1}{M'} \leqslant \frac{1}{M}.$$

If $c(M') > c(M)$ for no $M' > M$, then

$$\frac{1}{c(M)} \prod_{p \leqslant c(M)} \frac{p}{p-1} = \frac{1}{c(M')} \prod_{p \leqslant c(M')} \frac{p}{p-1} \leqslant \frac{1}{M'} \quad \text{ for all } M' > M,$$

and hence $c(M)^{-1} \prod_{p \leqslant c(M)} p/(p-1) = 0$ which is impossible. So $c(M) \to +\infty$ as $M \to +\infty$. By the definition of $c(M)$,

$$\frac{1}{c(M)} \prod_{p \leqslant c(M)} \frac{p}{p-1} \leqslant \frac{1}{M} < \frac{1}{c(M)-1} \prod_{p \leqslant c(M)-1} \frac{p}{p-1}.$$

Thus $c(M)$ cannot be a prime, and

$$1 - \frac{1}{c(M)} = \frac{c(M)-1}{c(M)} < \frac{M}{c(M)} \prod_{p \leqslant c(M)-1} \frac{p}{p-1} = \frac{M}{c(M)} \prod_{p \leqslant c(M)} \frac{p}{p-1} \leqslant 1.$$

Since $\prod_{p \leqslant c(M)}(1 - p^{-1}) \to 0$ as $M \to +\infty$, we have $M = o(c(M))$.
   By the above,

$$e^{\gamma} M \frac{\log c(M)}{c(M)} = \frac{M}{c(M)} \prod_{p \leqslant c(M)} \frac{p}{p-1} + O\left(\frac{M}{c(M)}\right)$$

$$= 1 + O\left(\frac{1}{c(M)}\right) + O\left(\frac{M}{c(M)}\right) = 1 + O\left(\frac{M}{c(M)}\right)$$

and hence

$$\frac{c(M)}{\log c(M)} = e^{\gamma} M \left(1 + O\left(\frac{M}{c(M)}\right)\right)^{-1} = e^{\gamma} M + O\left(\frac{M^2}{c(M)}\right).$$

It follows that

$$\log c(M) \sim \log \frac{c(M)}{\log c(M)} = \log(e^{\gamma} M(1 + o(1))) \sim \log M$$

and

$$c(M) \sim e^{\gamma} M \log c(M) \sim e^{\gamma} M \log M.$$

Thus

$$\log c(M) = \log(e^{\gamma} M \log M) + \log(c(M)/(e^{\gamma} M \log M))$$
$$= \gamma + \log M + \log \log M + o(1) = \log M + O(\log \log M)$$

and

$$c(M) = e^{\gamma} M \log c(M) + O\left(\frac{M^2 \log c(M)}{c(M)}\right)$$

$$= e^{\gamma} M (\log M + O(\log \log M)) + O\left(\frac{M^2}{e^{\gamma} M}\right)$$

$$= e^{\gamma} M \log M + O(M \log \log M).$$

The famous prime number theorem (see Chapter 4 of [Ap]) states that

$$\pi(x) = \sum_{p \leqslant x} 1 \sim \frac{x}{\log x} \quad \text{as } x \to +\infty,$$

moreover $\pi(x) = x/\log x + O(x/\log^2 x)$ (for $x \geqslant 2$) by [Bo] or [DV]. Hence

$$\pi(c(M)) = c(M)/\log c(M) + O(c(M)/\log^2 c(M))$$
$$= e^\gamma M + O\left(\frac{M^2}{e^\gamma M \log M}\right) + O\left(\frac{e^\gamma M \log M}{\log^2 M}\right) = e^\gamma M + O\left(\frac{M}{\log M}\right).$$

It is easy to see that

$$\frac{1}{l+1} \prod_{p \leqslant l+1} \frac{p}{p-1} \leqslant \frac{1}{l} \prod_{p \leqslant l} \frac{p}{p-1} \quad \text{for every } l = 1, 2, 3, \cdots.$$

Therefore

$$\frac{1}{n} \prod_{p \leqslant n} \frac{p}{p-1} \leqslant \frac{1}{c(M)} \prod_{p \leqslant c(M)} \frac{p}{p-1} \leqslant \frac{1}{M}$$

for all $n = c(M), c(M) + 1, \cdots$. When an integer $q > 1$ satisfies the inequality $q < M \prod_{p \leqslant q} p/(p-1)$ (i.e. $q^{-1} \prod_{p \leqslant q} p/(p-1) > M^{-1}$), we must have $q < c(M)$ and $\pi(q) \leqslant \pi(c(M))$, so (4.11) follows. This completes the proof. $\quad\square$

Theorem 4.3. *Let* (1.1) *be a nontrivial uniform cover of a group G such that among the indices $[G : G_1] \leqslant \cdots \leqslant [G : G_k]$ each occurs at most $M \in \mathbb{Z}^+$ times. Let p_* and p^* be the smallest and the largest prime divisors of $N = [[G : G_1], \ldots, [G : G_k]]$ respectively. Suppose that all the G_i with $[G : G_i] \geqslant p^*$ are subnormal in G, or G/H is a solvable group having a normal Sylow p'-subgroup where H is the largest normal subgroup of G contained in all the G_i and p' is the greatest prime divisor of $|G/H|$ (equivalently, there is a composition series from $H = (\bigcap_{i=1}^k G_i)_G$ to G whose quotients have prime order and if a quotient is not of the maximal order then neither is the next quotient). Then we have the following* (i)–(iv) *with the O-constants absolute.*

(i) *$M \geqslant p_*$, moreover among the k indices $[G : G_1], \ldots, [G : G_k]$ there exists a multiple of p^* occurring at least $1 + \lfloor p^* \prod_{p|N} (p-1)/p \rfloor \geqslant p_*$ times.*

(ii) *All prime divisors of $[G : G_1], \ldots, [G : G_k]$ are smaller than $e^\gamma M \log M + O(M \log \log M)$.*

(iii) *The number of distinct prime divisors of $[G : G_1], \ldots, [G : G_k]$ does not exceed $e^\gamma M + O(M/\log M)$.*

(iv) *For the least index, $\log[G : G_1] \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log \log M)$.*

Proof. Let $p_* = p_1 < \cdots < p_r = p^*$ be all the distinct prime divisors of N. By the supposition and Lemma 2.5, either all the G_i with $[G : G_i] \geqslant p_r$ are subnormal in G

and hence conditions (a) and (b) in Theorem 4.1 are satisfied, or we have condition (c) in Theorem 4.1. In light of Theorem 4.1,

$$p_r < \max_{\substack{1 \leqslant j \leqslant k \\ p_r \mid [G:G_j]}} |\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}| \prod_{t=1}^{r} \frac{p_t}{p_t - 1}.$$

So, for some $j = 1, \ldots, k$ with $[G : G_j]$ divisible by $p^* = p_r$, we have

$$|\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}|$$

$$> p_r \prod_{t=1}^{r} \frac{p_t - 1}{p_t} = p^* \prod_{p \mid N} \frac{p - 1}{p} = (p_r - 1) \prod_{t=1}^{r-1} \frac{p_t - 1}{p_t}$$

$$\geqslant p_{r-1} \prod_{t=1}^{r-1} \frac{p_t - 1}{p_t} \geqslant \cdots \geqslant p_1 \frac{p_1 - 1}{p_1} = p_* - 1$$

and hence $M \geqslant |\{1 \leqslant i \leqslant k \colon [G : G_i] = [G : G_j]\}| \geqslant 1 + \lfloor p^* \prod_{p \mid N} (p - 1)/p \rfloor \geqslant p_*$.

Note that $M > p^* \prod_{p \leqslant p^*} (p - 1)/p$. Let $c(M)$ be as in the proof of Lemma 4.2. By Lemma 4.2 and its proof, we have

$$p_1 < \cdots < p_r = p^* < c(M) = e^\gamma M \log M + O(M \log \log M)$$

and

$$r \leqslant \pi(p_r) = \pi(p^*) \leqslant \pi(c(M)) = e^\gamma M + O(M/\log M).$$

It is known that $\zeta(2) := \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. Let $\alpha(M) = 2 + \lfloor \log_2(\zeta(2)c(M)) \rfloor$. By induction, if $0 \leqslant x_1, \ldots, x_n \leqslant 1$ then $\prod_{i=1}^{n}(1 - x_i) \geqslant 1 - \sum_{i=1}^{n} x_i$. Thus

$$\prod_{p \leqslant c(M)} \left(1 - \frac{1}{p^{\alpha(M)+1}}\right) \geqslant 1 - \sum_{p \leqslant c(M)} \frac{1}{p^{\alpha(M)+1}}$$

$$\geqslant 1 - \sum_{p \leqslant c(M)} \frac{1}{p^2} \cdot \frac{1}{2^{\alpha(M)-1}} \geqslant 1 - \frac{\zeta(2)}{2^{\alpha(M)-1}} > 1 - \frac{1}{c(M)}$$

and hence

$$1 - \prod_{p \leqslant c(M)} \left(1 - \frac{1}{p^{\alpha(M)+1}}\right) < \frac{1}{c(M)} \leqslant \frac{1}{M} \prod_{p \leqslant c(M)} \frac{p - 1}{p}.$$

Therefore

$$\prod_{p \leqslant c(M)} \sum_{n=0}^{\infty} \frac{1}{p^n} - \frac{1}{M} = \prod_{p \leqslant c(M)} \frac{p}{p - 1} - \frac{1}{M}$$

$$< \prod_{p \leqslant c(M)} \frac{p}{p - 1} \cdot \prod_{p \leqslant c(M)} \left(1 - \frac{1}{p^{\alpha(M)+1}}\right)$$

$$= \prod_{p \leqslant c(M)} \left(\sum_{n=0}^{\infty} \frac{1}{p^n} - \frac{1}{p^{\alpha(M)+1}} \sum_{n=0}^{\infty} \frac{1}{p^n}\right) = \prod_{p \leqslant c(M)} \sum_{n=0}^{\alpha(M)} \frac{1}{p^n}.$$

If $w_{\mathcal{A}}(x) = m$ for all $x \in G$, then $\sum_{i=1}^{k} [G : G_i]^{-1} = m$ by Lemma 2.2 of [Su8]. Set $S = \{[G : G_1], \dots, [G : G_k]\}$ and let $T(M)$ be the set of positive integers which have no prime divisors greater than $c(M)$. Then

$$1 \leqslant \sum_{i=1}^{k} \frac{1}{[G : G_i]} \leqslant M \sum_{n \in S} \frac{1}{n} < M \sum_{\substack{n \in T(M) \\ n \geqslant [G:G_1]}} \frac{1}{n}$$

and thus

$$\sum_{\substack{n \in T(M) \\ n < [G:G_1]}} \frac{1}{n} = \sum_{n \in T(M)} \frac{1}{n} - \sum_{\substack{n \in T(M) \\ n \geqslant [G:G_1]}} \frac{1}{n} < \prod_{p \leqslant c(M)} \sum_{n=0}^{\infty} \frac{1}{p^n} - \frac{1}{M} < \prod_{p \leqslant c(M)} \sum_{n=0}^{\alpha(M)} \frac{1}{p^n}.$$

Now it is clear that $\prod_{p \leqslant c(M)} p^{\alpha(M)}$ cannot be less than $[G : G_1]$. So $[G : G_1] \leqslant \prod_{p \leqslant c(M)} p^{\alpha(M)}$ and hence

$$\log[G : G_1] \leqslant \alpha(M)\theta(c(M)) \leqslant l(M) := (2 + \log_2(\zeta(2)c(M))) \pi(c(M)) \log c(M)$$

where $\theta(x) = \sum_{p \leqslant x} \log p \ (\leqslant \pi(x) \log x)$ is the Chebyshev θ-function. By the proof of Lemma 4.2,

$$\log(\zeta(2)c(M)) = \log \frac{\pi^2}{6} + \log M + O(\log \log M) = \log M + O(\log \log M)$$

and

$$\begin{aligned}
\pi(c(M)) \log c(M) &= c(M) + O(c(M)/\log c(M)) \\
&= e^{\gamma} M \log M + O(M \log \log M) + O(e^{\gamma} M) \\
&= e^{\gamma} M \log M + O(M \log \log M).
\end{aligned}$$

So we finally have

$$\begin{aligned}
l(M) &= \frac{1}{\log 2} (\log M + O(\log \log M))(e^{\gamma} M \log M + O(M \log \log M)) \\
&= \frac{e^{\gamma}}{\log 2} M \log^2 M + O(M \log M \log \log M).
\end{aligned}$$

This concludes our proof. $\quad \square$

Remark 4.3. Obviously Theorem 4.3 provides more detailed information than Theorem 1.1 does.

For a nontrivial uniform cover (1.3) of $\mathbb{Z}$, it is known that among the k moduli the largest n_k occurs at least p times where p is the smallest prime divisor of n_k (cf. [Ne], [NZ], [Su3]). This, together with Theorem 4.3(i), suggests the following conjecture.

Conjecture 4.1. *Let* (1.1) *be a nontrivial uniform cover of a group G by left cosets of subnormal subgroups. Set $n = \max_{1 \leqslant i \leqslant k} [G : G_i]$. Then $|\{1 \leqslant i \leqslant k : [G : G_i] = n\}|$ is not less than the least prime divisor of n.*

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